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A note on multi-index polynomials of Dickson type and their applications in quantum optics

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Abstract

We discuss the properties of a new family of multi-index Lucas type polynomials, which are often encountered in problems of intracavity photon statistics. We develop an approach based on the integral representation method and show that this class of polynomials can be derived from recently introduced multi-index Hermite like polynomials. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

The two variable Hermite polynomials [1] $(n, m \geqslant 0)$

$$h_{m,n}(x, y; \tau) = m!n! \sum_{k=0}^{\min(m,n)} \frac{\tau^k x^{m-k} y^{n-k}}{k!(m-k)!(n-k)!} \tag{1}$$

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are defined through the generating function

$$\sum_{(m,n)=0}^{\infty} \frac{u^m}{m!}\frac{v^n}{n!} h_{m,n}(x,y;\tau) = \mathrm{e}^{xu+yv+\tau uv} \tag{2}$$

and the operational identity

$$h_{m,n}(x,y;\tau) = \mathrm{e}^{\tau \partial^2_{x,y}}(x^m y^n). \tag{3}$$

The polynomials

$$u_{m,n}(x,y;\tau) = \sum_{k=0}^{\min(m,n)} \frac{(m+n-k)!\tau^k x^{m-k} y^{n-k}}{k!(m-k)!(n-k)!} \tag{4}$$

often exploited in photon statistics [2], can be associated with $h_{m,n}(x,y;\tau)$ polynomials according to the identity

$$u_{m,n}(x,y;\tau) = \frac{1}{m!n!}\int_0^\infty \mathrm{d}t\mathrm{e}^{-t}t^{m+n}h_{m,n}\left(x,y;-\frac{\tau}{t}\right) \tag{5}$$

which can be exploited to infer the relevant generating function. Multiplying indeed both sides of Eq. (4) by u^m and v^n and then by summing up over the indices, we find (see also Eq. (2))

$$\sum_{(m,n)=0} u^m v^n u_{m,n}(x,y;\tau) = \frac{1}{1-ux-vy+\tau uv}. \tag{6}$$

This identity is particularly interesting if considered along with a previous result linking Hermite-like and second kind Tchebycheff polynomials [3]

$$U_n(x) = \frac{1}{n!}\int_0^\infty \mathrm{e}^{-t}t^n H_n\left(2x,-\frac{1}{t}\right)\mathrm{d}t, \tag{7}$$

where

$$H_n(x,y) = n!\sum_{r=0}^{[n/2]} \frac{y^k x^{n-2k}}{k!(n-2k)!} \tag{8}$$

and

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}H_n(x,y) = \mathrm{e}^{xt+yt^2}. \tag{9}$$

It is also been shown that the use of multi-variable forms of Hermite polynomials $H_n(\{x_s\})$, $\{x_s\} = x_1,\ldots,x_m$ with generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}H_n(\{x_s\}) = \mathrm{e}^{\sum_{s=1}^m x_s t^s} \tag{10}$$

allows to define Dickson-like polynomials [4] according to the identity

$$U_n(\{x_s\}) = \frac{1}{n!}\int_0^\infty \mathrm{d}t\mathrm{e}^{-t}t^n H_n\left(\left\{\frac{(-1)^{s-1}x_s}{t^{s-1}}\right\}\right). \tag{11}$$

yielding, by following a procedure analogous to that leading to Eq. (6), the generating function

$$\sum_{n=0}^{\infty} u^n U_n(\{x_s\}) = \frac{1}{1+\sum_{s=1}^{m} u^s(-1)^s x_s} \tag{12}$$

which coincides with that of Dickson polynomials.

It is evident that the $u_{m,n}(x,y;\tau)$ suggest the existence of multi-index polynomials generalizing the Dickson family and linked, by a suitable integral representation, to multi-variable Hermite polynomials. This paper is devoted to the study of the properties of these new families and to their applications in quantum optics.

2. Multi-index Dickson polynomials

The $u_{m,n}(x,y;\tau)$ polynomials can be considered as the simplest case of multi-index Dickson polynomial, notwithstanding they have fairly interesting and worth to be stressed properties.

We start from the generating function [5]

$$\sum_{(m,n)=0}^{\infty} \frac{u^m}{m!}\frac{v^n}{n!} h_{m+p,n+q}(x,y;\tau) = \mathrm{e}^{xu+yv-\tau uv} h_{p,q}(x-v\tau, y-u\tau;\tau) \tag{13}$$

which, according to Eq. (6), allows to write

$$\sum_{(m,n)=0}^{\infty} \frac{(m+p)!}{m!}\frac{(n+q)!}{n!} u_{m+p,n+q}(x,y;\tau)$$

$$= \int_0^{\infty} \mathrm{d}t \mathrm{e}^{-t} t^{p+q} \mathrm{e}^{(xu+yv-\tau uv)t} h_{p,q}\left(x-v\tau, y-u\tau; -\frac{\tau}{t}\right). \tag{14}$$

This last identity yields after a rescaling of the t variable and the use of Eq. (6)

$$\sum_{(m,n)=0}^{\infty} \frac{(m+p)!}{m!}\frac{(n+q)!}{n!} u_{m+p,n+q}(x,y;\tau)$$

$$= \frac{(p+q)!}{[1+ux+vy-\tau uv]^{p+q}} h_{p,q}(x-v\tau, y-u\tau; (1+ux+vy-\tau uv)\tau). \tag{15}$$

We can also attempt a further generalization of the polynomials $u_{m,n}(x,y;\tau)$ by introducing the quantities

$$u_{m,n}^{(\mu)}(x,y;\tau) = \frac{1}{\Gamma(\mu)m!n!}\int_0^{\infty} \mathrm{d}t \mathrm{e}^{-t} t^{\mu+n+m-1} h_{m,n}\left(x,y;-\frac{\tau}{t}\right) \tag{16}$$

whose generating function is

$$\sum_{(m,n)=0}^{\infty} \frac{u^m}{m!}\frac{v^n}{n!} u_{m,n}^{(\mu)}(x,y;\tau) = \frac{1}{[1+ux+vy-\tau uv]^{\mu}}, \tag{17}$$

satisfying the recurrences

$$\frac{\partial}{\partial x} u_{m,n}^{(\mu)}(x, y; \tau) = \mu u_{m-1,n}^{(\mu+1)}(x, y; \tau),$$

$$\frac{\partial}{\partial y} u_{m,n-1}^{(\mu)}(x, y; \tau) = \mu u_{m,n-1}^{(\mu+1)}(x, y; \tau),$$

$$\frac{\partial}{\partial \tau} u_{m,n}^{(\mu)}(x, y; \tau) = -\mu u_{m-1,n-1}^{(\mu+1)}(x, y; \tau) \tag{18}$$

and specified by the series

$$u_{m,n}^{(\mu)}(x, y; \tau) = \frac{1}{\Gamma(\mu)} \sum_{k=0}^{\min(m,n)} \frac{\Gamma(n+m+\mu-k)\tau^k x^{m-k} y^{n-k}}{k!(m-k)!(n-k)!}. \tag{19}$$

The key point of the present investigation is the use of integral representations, linking polynomials of Hermite nature with other families of polynomials which can be recognized as Dickson or Gegenbauer multi-index forms. It is, therefore, clear that the polynomials

$$H_{m,n}(x, y; z, w|\tau) = m!n! \sum_{s=0}^{\min(m,n)} \frac{\tau^s H_{m-s}(x, y) H_s(z, w)}{s!(m-s)!(n-s)!}, \tag{20}$$

with

$$H_s(x, y) = n! \sum_{r=0}^{[n/2]} \frac{y^s x^{n-2s}}{s!(n-2s)!} \tag{21}$$

and specified by the generating function

$$\sum_{(m,n)=0}^{\infty} \frac{u^m}{m!} \frac{v^n}{n!} H_{m,n}(x, y; z, w|\tau) = \mathrm{e}^{xu+yu^2+zv+wv^2+\tau uv} \tag{22}$$

define, the new family of two-index polynomials

$$U_{m,n}(x, y; z, w|\tau) = \frac{1}{m!n!} \int_0^{\infty} \mathrm{d}t \mathrm{e}^{-t} t^{m+n} H_{m,n}\left(x, -\frac{y}{t}; z, -\frac{w}{t}\middle| -\frac{\tau}{t}\right) \tag{23}$$

whose generating function

$$\sum_{(m,n)=0}^{\infty} u^m v^n U_{m,n}(x, y; z, w|\tau) = \frac{1}{1 - ux + u^2 y - vz + v^2 w + uv\tau} \tag{24}$$

can be obtained by following the already exploited procedure. It is also fairly straightforward to conclude that

$$U_{m,n}^{(\mu)}(x, y; z, w|\tau) = \frac{1}{\Gamma(\mu) m!n!} \int_0^{\infty} \mathrm{d}t \mathrm{e}^{-t} t^{m+n+\mu-1} H_{m,n}\left(x, -\frac{y}{t}; z, -\frac{w}{t}\middle| -\frac{\tau}{t}\right), \tag{25}$$

with

$$\sum_{(m,n)=0}^{\infty} u^m v^n U_{m,n}^{(\mu)}(x, y; z, w|\tau) = \frac{1}{[1 - ux + u^2 y - vz + v^2 w + uv\tau]^{\mu}}. \tag{26}$$

In this section, we have proved the possibility of exploiting families of multi-index Hermite polynomials to construct corresponding families of Gegenbauer or Dickson polynomials. In the forthcoming section, we will discuss more accurately the problem and analyze possible applications.

3. Concluding remarks

In the introductory remarks we have shown that the Dickson polynomials can be defined by exploiting Eq. (12), it is, therefore, natural to define further the polynomials

$$U_n^{(\mu)}(\{x_s\}) = \frac{1}{n!\Gamma(\mu)} \int_0^{\infty} \mathrm{d}t e^{-t} t^{n+\mu-1} H_n \left(\left\{ \frac{(-1)^{s-1} x_s}{t^{s-1}} \right\} \right), \tag{27}$$

with generating function

$$\sum_{n=0}^{\infty} u^n U_n^{(\mu)}(\{x_s\}) = \frac{1}{(1 + \sum_{s=1}^{m} (-u)^s x_s)^{\mu}}. \tag{28}$$

The use of the properties of the $H_n(\{x_s\})$ polynomials allows to derive those of $U_n^{(\mu)}(\{x_s\})$ in a way significantly simpler than those based on more conventional means. The use of the generalized Rainville identity [5]

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_{n+l}(\{x_s\}) = e^{P(\{x_s\},t)} H_l \left(\left\{ \frac{1}{r!} P^{(r)}(\{x_s\}, t) \right\} \right),$$

$$P(\{x_s\}, t) = \sum_{s=1}^{m} x_s t^s, \tag{29}$$

with $P^{(r)}(\{x_s\}, t)$ denoting the rth order derivative with respect to t, allows to state analogous results for the polynomials (27). We find, indeed (see also Ref. [3])

$$\sum_{n=0}^{\infty} u^n U_{n+l}^{(\mu)}(\{x_s\})$$

$$= \frac{1}{(1 + \sum_{s=1}^{m} (-u^s) x_s)^{\mu+l}} U_l^{(\mu)} \left(\left\{ \frac{1}{r!} P^{(r)}(\{x_s\}, u) \left(1 + \sum_{r=1}^{m} (-u)^r x_r \right)^{s-1} \right\} \right) \tag{30}$$

this result, new to the authors' knowledge, is a by-product of the integral representation and of the operational methods.

In Ref. [1], general families of multi-index Hermite polynomials have been introduced, as e.g., those defined by the generating functions

$$\sum_{(m,n)=0}^{\infty} \frac{u^m}{m!} \frac{v^n}{n!} H_{m,n}^{(3,3)}(\{x_\alpha\}, \{y_\alpha\} | \{\tau_{\alpha,.\beta}\}) = e^{\sum_{s=1}^{3} (x_s u^s + y_s v^s) + \tau_{1,1} uv + \tau_{2,1} u^2 v + \tau_{2,1} uv^2} \tag{31}$$

which can be exploited to define the polynomials

$$U_{m,n}^{(3,3)}(\{x_\alpha\},\{y_\alpha\}|\{\tau_{\alpha,\beta}\})$$
$$=\frac{1}{m!n!}\int_0^\infty \mathrm{d}t\mathrm{e}^{-t}t^{m+n}H_{m,n}^{(3,3)}\left(\left\{\frac{(-1)^{\alpha-1}x_\alpha}{t^{\alpha-1}}\right\},\left\{\frac{(-1)^{\beta-1}y_\alpha}{t^{\alpha-1}}\right\}\middle|\frac{(-1)^{\alpha+\beta-1}\tau_{\alpha,.\beta}}{t^{\alpha+\beta-1}}\right) \tag{32}$$

whose generating function can be obtained by following the already discussed procedure which yields

$$\sum_{(m,n)=0}^{\infty}\frac{u^m}{m!}\frac{v^n}{n!}U_{m,n}^{(3,3)}(\{x_\alpha\},\{y_\alpha\}|\{\tau_{\alpha,\beta}\})$$
$$=\frac{1}{1+\sum_{s=1}^{3}(-1)^s(x_su^s+y_sv^s)+\tau_{1,1}uv-\tau_{2,1}u^2v-\tau_{2,1}uv^2}. \tag{33}$$

It is also clear that we can extend the definition by including higher order terms, but the extension is fairly straightforward and is omitted for brevity's sake. Identities of the type (16) for the more general polynomials (32) will be discussed elsewhere.

As already remarked polynomials of the type discussed in this paper are currently exploited in optics, for example in problems associated with temporal correlations between photon intracavity detections, the propagation matrix $X_{m,n}$ for the probability distribution [6] is just a particular case of the $u_{m,n}(x,y;\tau)$ polynomials and indeed we get

$$X_{m,n}\propto u_{m,n}(1,1;\tau). \tag{34}$$

The use of the relevant generating function allows the evaluation of the various moments of the distribution in a direct way.

We have noted that the introduction of the polynomials $u_{m,n}(x,y;\tau)$ has been stimulated by the analogy with the Tchebycheff polynomials defined through the identity (7). As is well known the $U_n(x)$ are second kind Tchebycheff[1] while first kind have been shown to be provided by

$$T_n(x)=\frac{1}{2(n-1)!}\int_0^\infty \mathrm{d}t\mathrm{e}^{-t}t^{n-1}H_n\left(2x,-\frac{1}{t}\right) \tag{35}$$

we can, therefore, introduce the further polynomials

$$t_{m,n}^{(\alpha,\beta)}(x,y;\tau)=\frac{1}{(m-\alpha)!(n-\beta)!}\int_0^\infty \mathrm{d}t\mathrm{e}^{-t}t^{m+n-1}h_{m,n}\left(x,y;-\frac{\tau}{t}\right) \tag{36}$$

which, in analogy to the Tchebycheff case, satisfy the identities

$$\frac{\partial}{\partial x}t_{m,n}^{(1,0)}(x,y;\tau)=mu_{m-1,n}(x,y;\tau),$$
$$\frac{\partial}{\partial y}t_{m,n}^{(0,1)}(x,y;\tau)=nu_{m,n-1}(x,y;\tau),$$
$$\frac{\partial}{\partial \tau}t_{m,n}^{(1,1)}(x,y;\tau)=-mnu_{m-1,n-1}(x,y;\tau). \tag{37}$$

[1] See [7], and [2] and references therein.

The same consideration applies to the Dickson polynomials, we can indeed define

$$V_n^{(\alpha)}(\{x_s\}) = \frac{1}{(n-\alpha)!}\int_0^\infty \mathrm{d}t e^{-t} t^{n-1} H_n\left(\left\{\frac{(-1)^{s-1}x_s}{t^{s-1}}\right\}\right) \tag{38}$$

which according to the well-known property of the $H_n(\{x_s\})$ polynomials

$$\frac{\partial}{\partial x_r} H_n(\{x_s\}) = \frac{n!}{(n-r)!} H_{n-r}(\{x_s\}) \tag{39}$$

yields

$$\frac{\partial}{\partial x_r} V_n^{(r)}(\{x_s\}) = \frac{n!(-1)^{r-1}}{(n-r)!} U_{n-r}(\{x_s\}). \tag{40}$$

The polynomials $U_n(\{x_s\})$ and $V_n^{(r)}(\{x_s\})$ can also be viewed as multi-variable Tchebycheff polynomials (see for e.g., Ref. [8]), while the $u_{m,n}(\ldots)$ and $t_{m,n}^{(.,.)}(\ldots)$ their multi-index counterpart.

Before concluding the paper we will touch on a final point concerning the explicit expression of the polynomials introduced in this paper and given in integral form only. In the case of $U_n(\{x_s\})$ and limiting ourselves for simplicity to $s = 1, 2, 3$ and by recalling that

$$H_n(x_1, x_2, x_3) = n! \sum_{s=0}^{[n/3]} \frac{x_3^s H_{n-3s}(x_1, x_2)}{s!(n-3s)!} \tag{41}$$

we get

$$U_n(x_1, x_2, x_3) = \sum_{r=0}^{[n/3]} \frac{x_3^r}{r!} \sum_{s=0}^{[(n-3r)/2]} \frac{(n-2r-s)! x_2^s x_1^{n-3r-2s}}{s!(n-3r-2s)!}. \tag{42}$$

In a forthcoming investigation we will more deeply analyze in the theory of the so far introduced polynomials and discuss in detail their properties.

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References

[1] G. Dattoli, The Hermite Bessel and Laguerre Functions: a by-product of the monomiality principle, in: D. Cocolicchio, G. Dattoli, H.M. Srivastava (Eds.), Proceedings of the Workshop Advanced Special Functions and Applications, Melfi, Potenza, Italy, May 9–13, 1999, ARACNE, Roma, 1999.

[2] H.F. Arnoldus, Temporal correlation between photons detections from radiation in a cavity, J. Opt. B, to appear.

[3] G. Dattoli, S. Lorenzutta, C. Cesarano, From Hermite to Humbert Polynomials, Rend. Ist. Mat. Univ. Trieste, to appear.

[4] M. Bruschi, P.E. Ricci, I polinomi di Lucas e di Tchebycheff in più variabili, Rend. Mat. 13 (6) (1980) 507–530.

[5] L.E. Dickson, Linear Groups With an Exposition of Galois Field Theory, Dover, New York, 1958.
[6] G. Dattoli, S. Lorenzutta, G. Maino, A. Torre, Theory of multi-index multivariable Bessel functions and Hermite polynomials, Le Matematiche (Catania) 52 (1997) 179–197.
[7] L.C. Andrews, Special Functions For Engineers and Applied Mathematicians, MacMillan, New York, 1985.
[8] P.E. Ricci, I polinomi di Tchebycheff in più variabili, Rend. Mat. 11 (6) (1978) 295–327.